

LOWER BOUNDS FOR POSSIBLE SINGULAR SOLUTIONS FOR THE NAVIER-STOKES AND EULER EQUATIONS REVISITED.

JEAN C. CORTISSOZ AND JULIO A. MONTERO

ABSTRACT. In this paper we give optimal lower bounds for the blow-up rate of the $\dot{H}^s(\mathbb{T}^3)$ -norm, $\frac{1}{2} < s < \frac{5}{2}$, of a putative singular solution of the Navier-Stokes equations, and we also present an elementary proof for a lower bound on blow-up rate of the Sobolev norms of possible singular solutions to the Euler equations when $s > \frac{5}{2}$.

1. INTRODUCTION

The Navier-Stokes equations for an incompressible viscous fluid of viscosity $\nu = 1$ are given by

$$(NS) \quad \begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } X \times (0, \infty) \\ u(x, 0) = \psi, \quad \operatorname{div} u = 0. \end{cases}$$

where, in this paper, $X = \mathbb{R}^3$ or $X = \mathbb{T}^3$. It is known that given an initial condition of finite energy and which belongs to $\dot{H}^s(X)$ there is an interval of time $(0, \eta)$, $\eta > 0$, for which there is a unique smooth solution to (NS) in $C([0, T]; \dot{H}^s(X))$. Let then $T > 0$ be the largest $\eta > 0$ for which the unique solution with initial data $\psi \in \dot{H}^s(X)$ remains smooth. It is unknown whether $T < \infty$ or $T = \infty$. In the case that $T < \infty$, there is the interesting question of estimating a rate at which the \dot{H}^s -norm blows-up. In [5] the authors, based on ideas presented by Robinson, Sadowski and Silva in [11], showed an almost optimal lower bound for the blow-up rate of solutions of the Navier-Stokes equations with periodic boundary conditions on a bounded maximal interval of existence $(0, T)$, $T < \infty$, when this solution belongs to $\dot{H}^{\frac{3}{2}}(\mathbb{T}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{T}^3)$. To be more precise, it was shown that a regular solution of the Navier-Stokes equation whose maximal interval of existence (or regularity) is $(0, T)$, must satisfy

$$\|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \geq \frac{c}{\sqrt{(T-t)|\log(T-t)|}},$$

for a constant $c > 0$. In this paper we go a little further and give a proof of the expected optimal lower blow-up rate. Namely, we prove the following estimate on the blow-up rate of putative singular solutions to the Navier-Stokes equations:

$$\frac{C}{t^{\frac{1}{2}}} \leq \|u(T-t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)}, \quad C > 0.$$

The proof of this result requires a detailed inspection of the bounds on the non-linear term of the Navier-Stokes equations found in [11], and the application of an

2010 *Mathematics Subject Classification.* Primary 35Q30, Secondary 35B44.

Key words and phrases. Navier-Stokes, Euler, blow-up, homogeneous Sobolev spaces.

interpolation technique inspired by the method used by Hardy to prove Carlson's inequality (see [6]). We must add that this problem using different techniques has been treated in the papers [2] and [10].

The lower blow-up rates for putative singular solutions to the Navier-Stokes equations can be interpreted as a regularity criterion for solutions of the equation (as they give a lower bound on the size of the maximal interval of existence). These blow-up estimates were first stated for the L^p spaces, $p > 3$, without proof by Leray in his remarkable paper [9], and proved by Giga in [7] via semigroup theory. In this paper, we rather follow the elementary, and improve on, the proof on homogeneous Sobolev spaces given by Robinson, Sadowski and Silva for their blow-up estimates.

On the other hand, there exists the related problem of investigating the possible blow-up behavior of solutions to the incompressible Euler equations:

$$(E) \quad \begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 & \text{in } X \times (0, \infty) \\ u(x, 0) = \psi, \quad \operatorname{div}(u) = 0. \end{cases}$$

In fact, recently in a very nice paper [1], Chen and Pavlović showed the following (although they state their result in \mathbb{R}^3 and we do in \mathbb{T}^3 , our arguments apply in both cases, see Remark 1).

Theorem 1.1. *Let $u(x, t)$ be a solution of the periodic Euler equations in the class*

$$(1) \quad C^1([0, T], \dot{H}^{\frac{3}{2}+\delta}(\mathbb{T}^3)) \cap C([0, T], \dot{H}^{\frac{5}{2}+\delta}(\mathbb{T}^3)), \quad \delta > 0,$$

and let $T > 0$ be the minimum time for which u cannot be continued in the class (1). Then, there exists a finite, positive constant $C(\delta, \|u(0)\|_{L^2(\mathbb{T}^3)})$ such that

$$\|u(t)\|_{\dot{H}(\mathbb{T}^3)^{\frac{5}{2}+\delta}} \geq C(\delta, \|u(0)\|_{L^2(\mathbb{T}^3)}) \left(\frac{1}{T-t} \right)^{2+\frac{2}{5}\delta}.$$

The proof of this theorem given in [1] relies on obtaining a single exponential bound on the H^s norms of a solution of the Euler equations via a length parameter introduced by P. Constantin in [3]. In this paper we follow the approach suggested in [11] in conjunction with some ideas presented in [5], to give a less involved proof of Theorem 1.1.

Part of this paper was written while the second author was visiting the Mathematics Department at Cornell University, and he is quite grateful for their warm hospitality -and in particular to Prof. Tim Healy for his encouragement. He also must acknowledge the support of Colciencias and his home institution, the Universidad de los Andes for making this visit possible, and his advisor (the first named author of this paper) for his encouragement, and his almost always insightful observations. The first author wants to thank the second author for being a great student and colleague, and for all these wonderful years of shared mathematical enthusiasm. He also wants to thank his home institution, the Universidad de los Andes, for providing an excellent research environment and economic support (Proyecto Semilla P15.160322.009).

2. THE BLOW UP RATE FOR THE NAVIER-STOKES EQUATIONS

The next statement is essentially the same given in [11]. The main difference is that we show a proof which includes the case when the solution belongs to

$\dot{H}^{\frac{3}{2}}(\mathbb{T}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{T}^3)$. From now on, in this paper we shall use the notation

$$\|u\|_s := \|u\|_{\dot{H}^s(\mathbb{T}^3)},$$

and \hat{u}_k refers to the Fourier wavenumber of wavevector k of the function u .

Theorem 2.1. *Let $u(x, t) = (u_1, u_2, u_3)$ be a solution Navier-Stokes equations whose maximum interval of existence is $(0, T)$, $0 < T < \infty$, and such that $u \in C((0, T), \dot{H}^s(\mathbb{T}^3) \cap \dot{H}^{s+1}(\mathbb{T}^3))$, with $\frac{1}{2} < s < \frac{5}{2}$. Then the following estimate holds*

$$(2) \quad \frac{C_s}{t^{\frac{1}{2}(s-\frac{1}{2})}} \leq \|u(T-s)\|_{\dot{H}^s(\mathbb{T}^3)}.$$

Proof. First, we must recall the energy inequality found in [11]:

$$(3) \quad \frac{1}{2} \frac{d}{dt} (\|u(t)\|_s^2) + 4\pi^2 \|u(t)\|_{s+1}^2 \leq C_s \left(\sum_k |\hat{u}_k| |k|^r \right) \|u(t)\|_s \|u\|_{s+1-r},$$

with $0 \leq r \leq 1$. For the sake of completeness we will give a proof of this inequality below.

Now we pick $r = \frac{1}{2} \left(s - \frac{1}{2} \right)$, and apply the interpolation technique employed by Hardy in his proof of Carlson's inequality (see [6]), to the first factor on the right hand side of (3), to obtain:

$$\begin{aligned} \sum_k |\hat{u}_k| |k|^{\frac{1}{2}(s-\frac{1}{2})} &= \sum_k |\hat{u}_k| |k|^{\frac{1}{2}(s-\frac{1}{2})} \frac{\sqrt{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{5}{2}}}}{\sqrt{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{5}{2}}}} \\ &\leq \left(a \|u\|_s^2 + b \|u\|_{s+1}^2 \right)^{\frac{1}{2}} \left(\sum_k \frac{1}{a|k|^{s+\frac{1}{2}} + b|k|^{s+\frac{5}{2}}} \right)^{\frac{1}{2}} \\ &\leq \left(a \|u\|_s^2 + b \|u\|_{s+1}^2 \right)^{\frac{1}{2}} \left(\frac{4\pi}{\sqrt{ab}} \left(\frac{\sqrt{a}}{\sqrt{b}} \right)^{\frac{3}{2}-s} \int_0^\infty \frac{y^{\frac{3}{2}-s}}{1+y^2} dy \right)^{\frac{1}{2}}, \end{aligned}$$

if we choose $a = \|u(t)\|_{s+1}^2$ and $b = \|u(t)\|_s^2$ then the energy inequality (3) becomes

$$(4) \quad \frac{1}{2} \frac{d}{dt} (\|u(t)\|_s^2) + 4\pi^2 \|u(t)\|_{s+1}^2 \leq C_s \|u(t)\|_s^{\frac{s}{2}+\frac{3}{4}} \|u(t)\|_{s+1}^{\frac{5}{4}-\frac{s}{2}} \|u(t)\|_{\frac{s}{2}+\frac{5}{4}}.$$

Now, observe that $\frac{s}{2} + \frac{5}{4} = \left(\frac{s}{2} - \frac{1}{4} \right) s + \left(\frac{5}{4} - \frac{s}{2} \right) (s+1)$, so by interpolation between homogeneous Sobolev spaces, we get

$$\|u\|_{\frac{s}{2}+\frac{5}{4}} \leq \|u\|_s^{\frac{s}{2}-\frac{1}{4}} \|u\|_{s+1}^{\frac{5}{4}-\frac{s}{2}}.$$

Therefore, from inequality (4) we obtain

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_s^2) + 4\pi^2 \|u(t)\|_{s+1}^2 \leq C_s \|u(t)\|_s^{s+\frac{1}{2}} \|u(t)\|_{s+1}^{\frac{5}{2}-s}.$$

It is time to use Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, with the choice $p = \frac{2(s+\frac{1}{2})}{s-\frac{1}{2}}$ and $q = \frac{2}{\frac{5}{2}-s}$. We thus get

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_s^2) \leq c_s \left(\|u(t)\|_s^2 \right)^{\left(1 + \frac{1}{s-\frac{1}{2}}\right)}.$$

Finally, by integrating between $T - t$ and T the previous estimate, inequality (2) follows. \square

Remark 1. *Theorem 2.1 is also valid when we consider the case of the whole space, i.e., for solutions $u(t) \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{R}^3)$, this because all the calculations leading to its proof are valid on \mathbb{R}^3 if we change sums by integrals.*

As promised, we give a proof of inequality (3). It is a consequence of the following lemma (see the proof of Lemma 3.1 in [11]) which gives an estimate of the nonlinear term

$$|(B(u, u), u)_{\dot{H}^s}|,$$

where

$$B(u, u) = P(u \cdot \nabla u),$$

and P is the Leray projector.

Lemma 2.1. *For any $s > 1$ and $0 \leq r \leq 1$, we have*

$$\left| \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |k|^{2s} (k \cdot \hat{u}_{k-q}) (\hat{u}_q \cdot \bar{\hat{u}}_k) \right| \leq c_s \left(\sum_{k \in \mathbb{Z}^3} |k|^r |\hat{u}_k| \right) \|u\|_s \|u\|_{s+1-r},$$

for all $u \in \dot{H}^{s+1-r}(\mathbb{T}^3) \cap F^r$. Here \hat{u}_k denotes the Fourier wavenumber of u with wavevector k , and \bar{z} denotes the complex conjugate of z .

Proof. Since P is self-adjoint and $u(x, t)$ is divergence free, we have that (see [4], chapter 6 p. 53)

$$\sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |q|^s |k|^s (\hat{u}_{k-q} \cdot q) (\hat{u}_q \cdot \bar{\hat{u}}_k) = 0.$$

Using the inequality (see [8], p.39)

$$||x|^s - |y|^s| \leq s(2^s) |x - y| (|x - y|^{s-1} + |y|^{s-1}), \quad s > 1;$$

and the well know inequality

$$(|x| + |y|)^r \leq |x|^r + |y|^r \text{ if } 0 \leq r \leq 1,$$

we obtain:

$$\begin{aligned}
& \left| \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |k|^{2s} (k \cdot \widehat{u}_{k-q}) (\widehat{u}_q \cdot \overline{\widehat{u}}_k) \right| \\
&= \left| \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |k|^{2s} (k \cdot \widehat{u}_{k-q}) (\widehat{u}_q \cdot \overline{\widehat{u}}_k) - \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |q|^s |k|^s (\widehat{u}_{k-q} \cdot q) (\widehat{u}_q \cdot \overline{\widehat{u}}_k) \right| \\
&\leq \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |k|^s |(q \cdot \widehat{u}_{k-q})| |\widehat{u}_q| |\widehat{u}_k| ||k|^s - |q|^s| \\
&\leq s 2^{s-1} \sum_{k \in \mathbb{Z}^3} \sum_{q \in \mathbb{Z}^3} |k|^s |(q \cdot \widehat{u}_{k-q})| |\widehat{u}_q| |\widehat{u}_k| |k - q| (|k|^{s-1} + |q|^{s-1}) \\
&\leq s 2^s \sum_{q \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} |k|^s |k - q|^s |q| |\widehat{u}_{k-q}| |\widehat{u}_q| |\widehat{u}_k| \\
&\leq s 2^s \sum_{q \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} |k|^s |k - q|^s |q|^r |q|^{1-r} |\widehat{u}_{k-q}| |\widehat{u}_q| |\widehat{u}_k| \\
&\leq s 2^s \sum_{q \in \mathbb{Z}^3} \sum_{k \in \mathbb{Z}^3} |k|^s |k - q|^s |q|^r (|k - q|^{1-r} + |k|^{1-r}) |\widehat{u}_{k-q}| |\widehat{u}_q| |\widehat{u}_k| \\
&\leq s 2^{s+1} \sum_{q \in \mathbb{Z}^3} |q|^r |\widehat{u}_q| \sum_{k \in \mathbb{Z}^3} |k - q|^s |\widehat{u}_{k-q}| |k|^{s+1-r} |\widehat{u}_k| \\
&\leq s 2^{s+1} \left(\sum_{k \in \mathbb{Z}^3} |k|^r |\widehat{u}_k| \right) \|u\|_s \|u\|_{s+1-r},
\end{aligned}$$

which is what we wanted to prove. \square

The previous proof gives us also an lower bound on size on the maximal interval of existence. Indeed, the following result holds.

Corollary 2.1. *Let $u(x, t)$ be a solution of the Navier–Stokes equations with initial condition $u_0(x) \in \dot{H}^s(\mathbb{T}^3)$, $\frac{1}{2} < s < \frac{5}{2}$, and let $T > 0$ be the minimum time for blow-up. Then*

$$(5) \quad \frac{K_s}{(\|u_0\|_s)^{\frac{4}{2s-1}}} \leq T.$$

3. THE BLOW UP RATE FOR THE EULER EQUATIONS

Estimate (3) can be used to provide the promised elementary proof of Theorem 1.1. But before we present our proof we will need the following estimate.

Lemma 3.1. *Let*

$$\|u\|_{F^1} = \sum_{k \in \mathbb{Z}^3} |k| |\widehat{u}_k|.$$

There is a constant $c > 0$ which only depends on s such that

$$\|u\|_{F^1} \leq c \|u\|_{L^{\frac{s}{s-\frac{5}{2}}}(\mathbb{T}^3)}^{\frac{s-\frac{5}{2}}{s}} \|u\|_{\frac{5}{s}}^{\frac{5}{s}}.$$

Proof. We have that

$$\begin{aligned}
\sum_k |\hat{u}_k| |k| &= \sum_k |\hat{u}_k| |k| \frac{\sqrt{a+b|k|^{2s-2}}}{\sqrt{a+b|k|^{2s-2}}} \\
&\leq \left(\sum_k |\hat{u}_k|^2 |k|^2 (a+b|k|^{2s-2}) \right)^{\frac{1}{2}} \left(\sum_k \frac{1}{a+b|k|^{2s-2}} \right)^{\frac{1}{2}} \\
&\leq c_s \left(a \|u\|_1^2 + b \|u\|_s^2 \right) \left(\int_0^\infty \frac{x^2 dx}{a+bx^{2s-2}} \right)^{\frac{1}{2}} \\
&\leq c_s \left(a \|u\|_1^2 + b \|u\|_s^2 \right) \frac{1}{\sqrt{a}} \left(\frac{a}{b} \right)^{\frac{3}{2(2s-2)}}.
\end{aligned}$$

We let $a = \|u\|_s^2$ and $b = \|u\|_1^2$ to obtain (for a new constant $c_s > 0$)

$$\begin{aligned}
\|u\|_{F^1} &\leq c_s \|u\|_1 \left(\frac{\|u\|_s}{\|u\|_1} \right)^{\frac{3}{2s-2}} \\
&= c_s \|u\|_1^{\frac{2s-5}{2s-2}} \|u\|_s^{\frac{3}{2s-2}}.
\end{aligned}$$

Now we use the Sobolev interpolation inequality

$$\|u\|_1 \leq \|u\|_{L^2(\mathbb{T}^3)}^{\frac{s-1}{s}} \|u\|_s^{\frac{1}{s}},$$

to obtain

$$\begin{aligned}
\|u\|_{F^1} &\leq c_s \|u\|_{L^2(\mathbb{T}^3)}^{\frac{2s-5}{2s}} \|u\|_s^{\frac{1}{s} \frac{2s-5}{2s-2} + \frac{3}{2s-2}} \\
&= c_s \|u\|_{L^2(\mathbb{T}^3)}^{\frac{2s-5}{2s}} \|u\|_s^{\frac{5}{2s}}.
\end{aligned}$$

□

We are ready to prove the estimate of Chen and Pavlović. Indeed, proceeding as we did in the proof of Theorem 2.1, for the Euler equations we obtain (i.e., by inequality (3) with $r = 1$; the extra positive term on the left-hand side does not appear, due to the lack of the diffusion term in the Euler equations) :

$$\frac{d}{dt} \|u\|_s^2 \leq c_s \|u\|_{F^1} \|u\|_s^2,$$

(this is just equation (6.2) in [11]), and hence by the previous lemma we arrive at the differential inequality

$$\frac{d}{dt} \|u\|_s^2 \leq c_s \|u\|_{L^2(\mathbb{T}^3)}^{\frac{2s-5}{2s}} \|u\|_s^{\frac{5}{2s}+2}.$$

For a regular solution to Euler equation, it is well-known that for any $t \geq 0$,

$$\|u(t)\|_{L^2(\mathbb{T}^3)} \leq \|u(0)\|_{L^2(\mathbb{T}^3)},$$

so we obtain an inequality (here the constant involved depends on $\|u(0)\|_{L^2(\mathbb{T}^3)}$)

$$\frac{d}{dt} \|u\|_s^2 \leq c \left(\|u(0)\|_{L^2(\mathbb{T}^3)}, s \right) \|u\|_s^{\frac{5}{2s}+2}.$$

Let $s = \frac{5}{2} + \delta$. Then our inequality becomes

$$\frac{d}{dt} \|u\|_{\frac{5}{2}+\delta}^2 \leq c \left(\|u(0)\|_{L^2(\mathbb{T}^3)}, \delta \right) \|u\|_{\frac{5}{2}+\delta}^{2+\frac{1}{1+\frac{2}{5}\delta}}.$$

Integrating the previous differential inequality from t to T (and assuming blow up at T) we get

$$\frac{1}{\|u(t)\|_{\frac{1}{\frac{1}{\frac{1}{2}+\delta}}}} \leq c(T-t),$$

where c is a constant that only depends on $\|u(0)\|_{L^2(\mathbb{T}^3)}$ and δ . Solving for $\|u(t)\|_{\frac{1}{\frac{1}{2}+\delta}}$ finishes the proof.

Remark 2. As commented before in the case of the Navier-Stokes equations, in the proofs in this section it is possible to replace \mathbb{T}^3 by \mathbb{R}^3 .

4. FINAL COMMENTS: SOME OPEN QUESTIONS

Theorem 2.1 includes the optimal lower bound for blow-up rates when $u \in \dot{H}^{\frac{3}{2}}(\mathbb{T}^3) \cap \dot{H}^{\frac{5}{2}}(\mathbb{T}^3)$; this particular case was missing in the proof given in [11], and in [5] a non optimal bound was proved. These bounds raise the following question: If there exists some $C > 0$ such that $\|u(T-t)\|_s \leq Ct^{-\frac{1}{2}(s-\frac{1}{2})}$, does $\|u(T-t)\|_s$ blow-up? Furthermore, a lower blow-up rate for $u \in \dot{H}^{\frac{1}{2}}(\mathbb{T}^3)$, for putative blow-up solutions to the Navier-Stokes equations, is yet unknown.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, BOGOTÁ DC, COLOMBIA